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ON THE STRUCTURE OF IDEALS OF THE DUAL ALGEBRA OF A COALGEBRA

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ABSTRACT. The weak-* topology is seen to play an important role in the study of various finiteness conditions one may place on a coalgebra C and its dual algebra C^*. Here we examine the interplay between the topology and the structure of ideals of C^*. The basic theory has been worked out for the commutative and almost connected cases (see [2]). Our basic tool for reducing to the almost connected case is the classical technique of lifting idempotents. Any orthogonal set of idempotents modulo a closed ideal of Rad C^* can be lifted. This technique is particularly effective when $C = C_1$. The strongest results we obtain concern ideals of C_1^*. Using the properties of idempotents we show that $C_1 = \Sigma_{x,y} C_x \wedge C_y$ where C_x and C_y run over the simple subcoalgebras of C. Our first theorem states that a coalgebra C is locally finite and C_0 is reflexive if and only if every cofinite ideal of C^* contains a finitely generated dense ideal. We show in general that a cofinite ideal I which contains a finitely generated dense ideal is not closed. (In fact either equivalent condition of the theorem does not imply C reflexive.) The preceding statement is true if $C = C_1$, or more importantly if $I \supset$ Rad C^* and C^*/I is algebraic. The second theorem characterizes the closure of an ideal with cofinite radical which also contains a finitely generated dense ideal.

In this paper we examine some of the basic aspects of the relationship between the weak-* topology and the structure of ideals of the dual algebra of a coalgebra over a field. In [2] and [4] the weak-* topology is seen to play an important role in the study of various finiteness conditions that one may place on a coalgebra. Here we try to put the topological ideas developed in these two references into a more general context.

Regarding a coalgebra C as a C-bicomodule we turn it into a left $\mathcal{C} = C \otimes C^{\mathrm{op}}$-comodule. This makes C^* a left $\Rightarrow\mathcal{C}^*$-module.The cyclic submodules are the closures of the principal (two-sided) ideals of C^*. Thus $\mathcal{C}^*$-submodules are ideals, and the finitely generated submodules are the closed ideals of C^* which contain a finitely generated dense ideal (see [2]). If $\mathcal{M}$ is a cofinite maximal ideal of C^* or more generally an algebraic ideal which contains the Jacobson radical Rad C^*, then $\mathcal{M}$ is a $\mathcal{C}^*$-submodule (see [4]). A minor theme of this paper is the connection between the ideals and $\mathcal{C}^*$-submodules.

If all ideals of the C^* are closed then C must be of finite type. For any coalgebra C the finite-dimensional rational C^*-modules are those which have closed cofinite annihilator. If C is reflexive (all cofinite ideals closed) then C is locally finite and C_0 is reflexive (see [2]). We show that the latter condition is

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equivalent to saying that all cofinite C^*-submodules of C^* are closed, or that each cofinite ideal contains a finitely generated dense ideal.

To require C to be locally finite and C_0 reflexive is not enough to insure that C is reflexive unless $C = C_1$. If $C = C_1$ then a cofinite ideal I of C^* which contains a finitely generated dense ideal must be closed. This is not the case in general, even if $C = C_2$.

The classical technique of lifting idempotents seems to be a very fruitful way of studying C_1. We show that for a coalgebra C any orthogonal set of idempotents modulo a closed ideal of Rad C^* can be lifted. Cofinite ideals under fairly general circumstances contain nonzero idempotents. Our most useful result is when $C = C_1$ and $e^2 = e \in C^*$, the (e) is a closed ideal provided e (restricted to C_0) is central.

For a coalgebra C write $C_0 = \coprod C_x$ as the direct sum of simple coalgebras. By the methods cited above we show $C_1 = \sum_{x,y} C_x \wedge C_y$ (see also [7]).

In §4 we characterize the closure of certain ideals in terms of $\mathcal{R} =$ Rad C^*. We first show that an ideal I which contains a finitely generated dense ideal and such that $\mathrm{Rad}(C^*/I)$ is cofinite is closed if $C/I^\perp$ is a semisimple C-comodule. From this we show that the closure of an ideal I which contains a finitely generated dense ideal such that $\mathrm{Rad}(C^*/I)$ is cofinite is $\mathcal{R} \cdot I$.

Throughout this paper we shall freely use the notation and results of [2] and [4].

1. Preliminaries.

1.1. *C^* as a topological space.* For convenience we isolate those topological results found in [2] which will be needed here.

Let V be a vector space. Throughout this paper we shall regard V^* as a topological space with the weak-* topology. The closed subspaces are the annihilators $W^\perp$ of subspaces W of V. For a subspace I of V^* let $I^\perp$ denote the annihilator of I in V; an important observation is that $\bar{I} = I^{\perp\perp}$. We say that a subspace J contained in I is *dense* (in I) if $\bar{J} = \bar{I}$.

Let $f\colon V \to W$ be linear. Then $f^*\colon W^* \to V^*$ is continuous, and more importantly,

1.1.1. $f^*\colon W^* \to V^*$ is *linearly closed*, meaning $f^*(I)$ is closed whenever I is a closed subspace of W^*.

Suppose $I, J \subset W^*$ are subspaces such that $\bar{I} = \bar{J}$. Since f^* is continuous $\overline{f^*(I)} = \overline{f^*(J)}$. Combining this observation with 1.1.1 we have

1.1.2. $f^*(\bar{I}) = \overline{f^*(I)}$ for any subspace I of W^*.

Let N be a left C-comodule. Then N^* is a left C^*-module. For a fixed $n^* \in N^*$ the map $L_{n^*}\colon C^* \to N^*$ defined by $L_{n^*}(c^*) = c^* n^*$ is continuous and linearly closed ($L_{n^*} = f^*$ where $f\colon N \to C$ is defined by $f(n) = \sum n_{(1)}\langle n^*, \underline{n}_{(2)}\rangle$). Since the sum of closed subspaces is closed, finitely generated submodules of N^* are closed.

The focus of this whole discussion is the following basic example.

1.1.3. Let C be a coalgebra and C^{op} denote the coalgebra C with "twisted" structure, set $\mathcal{C} = C \otimes C^{op}$. Then (C, ω) is a left $\mathcal{C}$-module ($\omega(c) = \sum c_{(1)} \otimes c_{(3)} \otimes c_{(2)} \in \mathcal{C} \otimes C$). Now $\mathcal{A} = C^* \otimes C^{*op}$ is a dense subalgebra of $\mathcal{C}^*$ and the module action on C^* restricted to $\mathcal{A}$ is determined by $a \otimes b \cdot c^* = ac^*b$ for $a \otimes b \in \mathcal{A}$ and $c^* \in C^*$. Thus, by 1.1.2,

1.1.4. $\mathcal{C}^* \cdot a = \overline{\mathcal{A} \cdot a} = \overline{(a)}$, where (a) is the principal (two-sided) ideal generated by $a \in C^*$.

This means that $\mathcal{C}^*$-submodules of C^* are ideals, and that closed ideals of C^* are $\mathcal{C}^*$-submodules.

Furthermore (1.3.5 of [2]),

1.1.5. the finitely generated $\mathcal{C}^*$-modules of C^* are the closed ideals which contain a finitely generated dense ideal.

1.2. *Semisimple C^*-modules.* The ideas in this section are of relevance primarily to §4.

For a coalgebra C let $R(C) = C/C_0^+$ where $C_0^+ = C_0 \cap \ker \varepsilon$. Then $R(C)$ is connected, and moreover the coradical filtration of each is related in a natural way. For if $\pi: C \to R(C)$ is the natural projection then

1.2.1. $\pi(C_n) = R(C)_n$ for all $n \geq 0$.

It is shown in [2] that $C_0^\perp = \text{Rad } C^*$, the Jacobson radical of C^*. Thus,

1.2.2. $R(C)^* = \text{Rad } C^* \oplus k \cdot \varepsilon$.

Notation. If C is a coalgebra and $\mathcal{C} = C \otimes C^{op}$, set $\mathcal{J} = \text{Rad}\, \mathcal{C}^*$ and $\mathcal{R} = R(\mathcal{C})^*$.

1.2.3. Lemma. *Let N be a simple left C-comodule. Then N is a D-comodule for some simple subcoalgebra D of C.*

Proof. Since N is simple, $N = n \leftharpoonup C^*$ for any fixed $0 \neq n \in N$. But I_n (the annihilator of n) is a cofinite maximal right ideal of C^*. Thus the annihilator $\mathcal{M}$ of N is a cofinite primitive (hence maximal) ideal of C^*. Let $D = \mathcal{M}^\perp$. Q.E.D.

For a left C-comodule N let N_0 denote the sum of the simple subcomodules of N.

1.2.4. Lemma. *Let N be a left C-comodule and $J = \text{Rad } C^*$. Then $N_0 = \{n \in N: n \leftharpoonup J = (0)\}$, so $N_0 = (J \cdot N^*)^\perp$.*

Proof. $M = \{n \in N: n \leftharpoonup J = (0)\}$ is a submodule of N, in fact a C_0-comodule since $J^\perp = C_0$. By 14.0.1, p. 288 of [6], $M \subset N_0$. By 1.2.3, $N_0 \subset M$. The rest is clear. Q.E.D.

For a left C^*-module M let Rad M denote the intersection of all maximal submodules of M.

1.2.5. Proposition. *Suppose N is a left C-comodule and N^* is a finitely generated C^*-module. If $J = \text{Rad } C^*$, then*

(1) $\text{Rad } N^* = J \cdot N^*$ and

(2) $0 \to J \cdot N^* \to N^* \to N_0^* \to 0$ is exact.

Proof. $J \cdot N^*$ is closed since J is a closed ideal and N^* is finitely generated. Thus $J \cdot N^* = (J \cdot N^*)^{\perp\perp} = N_0^\perp$. The result is now immediate. Q.E.D.

1.2.6. Corollary. *Suppose N is a left C-comodule and M is a finitely generated submodule of N^*. Then* Rad $M = J \cdot M$.

Proof. M is closed, so the projection $N \to N/M^{\perp} = Q$ induces an inclusion of C^*-modules $Q^* \to^u N^*$ with $u(Q^*) = M$. By the previous proposition Rad $M = u(\text{Rad } Q^*) = u(J \cdot Q^*) = J \cdot M$. Q.E.D.

2. Ideals and $\mathcal{C}^*$-submodules. In this section we examine the relationship between ideals and the $\mathcal{C}^*$-submodules of C^*. It turns out that the classical technique of lifting idempotents is particularly useful in this regard when $C = C_1$.

2.1. *C as a $\mathcal{C}$-comodule.* In Example 1.1.3, we introduced the left $\mathcal{C} = C \otimes C^{\text{op}}$-comodule action on a coalgebra C. If N is a C-bicomodule (a left and right C-comodule with commuting actions) then the map ω: $N \to \mathcal{C} \otimes N$ ($\omega(n) = \sum n_{(1)} \otimes n_{(3)} \otimes \underline{n}_{(2)}$) turns N into a left $\mathcal{C}$-comodule; and the subbicomodules (C^*-subbimodules) are the left $\mathcal{C}$-subcomodules. The following statements are immediate.

2.1.1. The category of left $\mathcal{C}$-comodules and the category of C-bicomodules are isomorphic.

2.1.2. The subcoalgebras of C (C^*-subbimodules) are the $\mathcal{C}$-subcomodules.

2.1.3. Δ: $C \to \mathcal{C}(\Delta c = \sum c_{(1)} \otimes c_{(2)})$ is a map of $\mathcal{C}$-comodules.

2.1.4. Suppose π: $C \to D$ is a map of coalgebras. Then $\Pi = \pi \otimes \pi$: $\mathcal{C} \to \mathfrak{D}(\mathfrak{D} = D \otimes D^{\text{op}})$ is a map of coalgebras. Thus (by pushout) C is a left $\mathfrak{D}$-comodule. The algebra map Π^*: $\mathfrak{D}^* \to \mathcal{C}^*$ induces (by pull-back) a left $\mathfrak{D}^*$-module structure on C^*. This is the transpose of the $\mathfrak{D}$-comodule structure on C.

3.17 of [4] gives the first important connection between ideals and $\mathcal{C}^*$-modules.

For an ideal I of C^* let Rad(I) denote the intersection of all maximal ideals lying above I. Recall that a k-algebra A is called *algebraic* if $k[a]$ is finite dimensional for all $a \in A$, and we call an ideal I of A algebraic if A/I is an algebraic algebra.

2.1.5. Proposition. (3.17 of [4]). *Suppose C is any coalgebra and I an algebraic ideal of C^* containing* Rad C^*. *Then I is a $\mathcal{C}^*$-module.*

The following elementary observation is useful.

2.1.6. Suppose C is a coalgebra and I is a closed ideal of C^*. Then Rad(I) is closed.

Proof. If $D = I^{\perp}$ then $0 \to I \to C^* \to^u D^* \to 0$ is exact, where u is the restriction map. It is clear that Rad$(I) = u^{-1}(\text{Rad } D^*)$. But Rad D^* is closed, thus Rad(I) is also since u is continuous. Q.E.D.

2.1.7. Proposition. *Suppose I is an ideal of C^* which contains a finitely generated dense ideal. If all maximal ideals which contain I are algebraic, then* Rad(I) *is closed.*

Proof. Suppose $(a_1, \ldots, a_n)$ is dense in I. Then $\bar{I} = \mathcal{C}^* \cdot a_1 + \cdots + \mathcal{C}^* \cdot a_n$

$\subset \mathcal{C}^* \cdot I \subset \bar{I}$ implies $\bar{I} = \mathcal{C}^* \cdot I$. If $\mathcal{M}$ is a maximal ideal which contains I, then $\bar{I} = \mathcal{C}^* \cdot I \subset \mathcal{M}$ by 2.1.5. Thus $\mathrm{Rad}(I) = \mathrm{Rad}(\bar{I})$ and the conclusion follows by 2.1.6. Q.E.D.

2.1.8. Proposition. *Suppose I contains a finitely generated dense ideal. Then I is dense (in C^*) if and only if no proper algebraic ideal contains I.*

Proof. ($\Rightarrow$) If $I \subset J$ where J is a proper algebraic ideal, then there is a maximal ideal $\mathcal{M}$ lying over J, and $\mathcal{M}$ is necessarily algebraic. Thus $\bar{I} = \mathcal{C}^* \cdot I \subset \mathcal{M}$ by 2.1.5, so $\bar{I} \neq C^*$.

($\Leftarrow$) Let $D = I^{\perp}$. Then we have the exact sequence $0 \to \bar{I} \to C^* \to D^* \to 0$. Thus $D = (0)$; otherwise there is a proper cofinite maximal ideal lying over I. Q.E.D.

In §2.2, we will show that when $C = C_1$ a cofinite ideal of C^* contains a cofinite $\mathcal{C}^*$-submodule under fairly general circumstances. An essential step of the argument depends on Corollary 2.1.10 of the following statement:

2.1.9. Proposition. *Suppose $\pi\colon C \to D$ is a surjective map of coalgebras with* $\ker \pi$ *finite dimensional. Then every cofinite ideal of C^* contains a cofinite $\mathcal{C}^*$-submodule if and only if every cofinite ideal of D^* contains a cofinite $\mathcal{D}^*$-submodule.*

Proof. The induced algebra map $u = \pi^*\colon D^* \to C^*$ is injective and $A = \pi^*(D^*)$ is a cofinite subalgebra of C^*.

($\Rightarrow$) Suppose I is a cofinite ideal of D^*. Then $u(I)$ is a cofinite ideal of A. Thus $M = C^*/u(I)$ is a finite-dimensional left A-module, so $JC^* \subset u(I)$ for some cofinite ideal J of A. But JC^* is a cofinite right ideal of C^*; thus the annihilator L of the right C^*-module C^*/JC^* is a cofinite ideal of C^* and $L \subset JC^* \subset u(I)$. By assumption there is a cofinite $\mathcal{C}^*$-module $K \subset L$. Therefore $u^{-1}(K)$ is a $\mathcal{D}^*$-submodule of D^* and $u^{-1}(K) \subset u^{-1}(u(I)) = I$.

($\Leftarrow$) Let I be a cofinite ideal of C^*. Then $J = u^{-1}(I)$ is a cofinite ideal of D^*. By assumption there is a cofinite $\mathcal{D}^*$-submodule $J_1 \subset J$. Since u is also a map of $\mathcal{D}^*$-modules by 2.1.4, $I_1 = u(J_1)$ is a cofinite $\mathcal{D}^*$-submodule of C^*. Now $\Delta^*\colon \mathcal{C}^* \to C^*$ is a map of $\mathcal{C}^*$-modules by 2.1.3, hence a map of $\mathcal{D}^*$-modules. Therefore $\gamma_1 = \Delta^{*-1}(I_1)$ is a cofinite $\mathcal{D}^*$-submodule of $\mathcal{C}^*$. Since $A = u(D^*)$ has finite codimension, the dense subalgebra $C^* \otimes C^{*\mathrm{op}}$ of $\mathcal{C}^*$ is a finitely generated left $D^* \otimes D^{*\mathrm{op}}$-module. Therefore, by 2.1.4 again, $\mathcal{C}^*$ is a finitely generated left $\mathcal{D}^*$-module. Now $M = \mathcal{C}^*/\gamma_1$ is a finite-dimensional left $\mathcal{C}^*$-module, so $\mathcal{L}\mathcal{C}^* \subset \gamma_1$ for some cofinite ideal $\mathcal{L}$ of $\mathcal{D}^*$. By 1.3.8 of [3], the right ideal $\mathcal{L}\mathcal{C}^*$ of $\mathcal{C}^*$ is cofinite. Hence there is a cofinite ideal γ of $\mathcal{C}^*$ (see the preceding paragraph) with $\gamma \subset \mathcal{L}\mathcal{C}^*$. $\Delta^*(\gamma)$ is a cofinite $\mathcal{C}^*$-submodule of C^*, and $\Delta^*(\gamma) \subset \Delta^*(\gamma_1) \subset I_1 = u(J_1) \subset I$. Q.E.D.

2.1.10. Corollary. *Suppose $C = C_1$ and is almost connected. Then every cofinite ideal of $\mathcal{C}^*$ contains a cofinite $\mathcal{C}^*$-submodule.*

Proof. The projection $C \xrightarrow{\pi} R(C)$ is surjection and $\ker \pi = C_0^+$ is finite dimensional. If $J = \mathrm{Rad}\, C^*$ then $J^2 = (0)$. By 1.2.2, $R(C)^* = J \oplus k \cdot 1$ so $R(C)^*$ is commutative.

The result follows easily from 1.1.4 and the preceding proposition, since principal left ideals are closed. Q.E.D.

2.2. *On lifting idempotents.* Let C be any coalgebra. We first show that any orthogonal set of idempotents modulo a closed ideal $I \subset \operatorname{Rad} C^*$ can be lifted to an orthogonal set of idempotents.

2.2.1. Lemma. *Suppose C is a finite-dimensional coalgebra, D a subcoalgebra with $C_0 \subset D$. If $\{e_1, \ldots, e_n\} \subset D^*$ is an orthogonal set of nonzero idempotents, there exists an orthogonal set of idempotents $\{u_1, \ldots, u_n\} \subset C^*$ with $u_i \equiv e_i$ on D for all $1 \leq i \leq n$.*

Proof. Let $I = D^\perp$. Then $I \subset C_0^\perp = \operatorname{Rad} C^*$. We have the exact sequence $0 \to I \to C^* \to^u D^* \to 0$ where u is the restriction map. Choose $a_1, \ldots, a_n \in C^*$ with $u(a_i) = e_i$ for all $1 \leq i \leq n$. Then $\{a_1, \ldots, a_n\}$ is an orthogonal set of nonzero idempotents modulo I, so, by Proposition 2 on p. 73 of [1], there exists an orthogonal set of idempotents $u_1, \ldots, u_n \in C^*$ with $u_i - a_i \in I$, i.e., $u_i \equiv e_i$ on D for all $1 \leq i \leq n$. Q.E.D.

2.2.2. Proposition. *Let C be any coalgebra and D a subcoalgebra with $D \supset C_0$. If $\{e_\alpha\} \subset D^*$ is an orthogonal set of idempotents then there exists an orthogonal set of idempotents $\{u_\alpha\} \subset C^*$ with $u_\alpha \equiv e_\alpha$ on D for all α.*

Proof. Let $\mathcal{S}$ be the family of all pairs $(E, \{v_\alpha\})$ where E is a subcoalgebra such that $D \subset E$, and $\{v_\alpha\} \subset E^*$ is an orthogonal set of idempotents satisfying $v_\alpha \equiv e_\alpha$ on D. Then $\leq$ defines a partial order of $\mathcal{S}$, where $(E, \{v_\alpha\}) \leq (F, \{w_\alpha\})$ denotes $E \subset F$ and $w_\alpha \equiv v_\alpha$ on E all α. Zorn's lemma applies; thus there is a maximal element $(E, \{w_\alpha\}) \in \mathcal{S}$.

Suppose F is a finite-dimensional subcoalgebra of C. Then $F_0 = C_0 \cap F \subset E \cap F$. Since nonzero orthogonal idempotents are independent, the set $\{v'_\alpha\}$ has only finitely many nonzero elements (v'_α is v_α restricted to $E \cap F$). By 2.2.1, there exists an orthogonal set of idempotents $\{w_\alpha\} \subset F^*$ with $w_\alpha \equiv v_\alpha$ on $E \cap F$. It is easy to see that the rule $u_\alpha(e + f) = v_\alpha(e) + w_\alpha(f)$ defines a functional $u_\alpha \in (E + F)^*$ for all α and that $(E + F, \{u_\alpha\}) \in \mathcal{S}$. This implies $E = C$. Q.E.D.

An immediate consequence of 2.2.2 is the following:

2.2.3. Let C be a coalgebra, D a subcoalgebra such that $C_0 \subset D$, and $e^2 = e \in D^*$. There exists an idempotent $u \in C^*$ with $u \equiv e$ on D.

2.2.4. Corollary. *Suppose C is a coalgebra and I is a cofinite ideal of C^* with* $\operatorname{Rad}(I)$ *closed. There is an idempotent $e \in I$ which is central (when restricted to C_0) and such that* $\operatorname{Rad}(I) = \operatorname{Rad}((e))$.

Proof. We have the exact sequence $0 \to \operatorname{Rad} C^* \to C^* \to^u C_0^* \to 0$ where u is the restriction map. Since $J = \operatorname{Rad}(I)$ is closed, $u(J)$ is closed. By 3.5.1 of [2], $u(\operatorname{Rad}(I)) = (f)$ for some central idempotent $f \in C_0^*$. By 2.2.3, there is an idempotent $e \in C^*$ with $u(e) = f$. Thus $e \in J = u^{-1}(u(J))$, and since $J^n \subset I$

some n, we see that $e = e^n \in I$. It is clear that $\mathrm{Rad}((e)) = J$. Q.E.D.

2.2.5. Remark. Suppose $\{e_\alpha\}$ is *any* orthogonal set of idempotents of C^*. Then $e = \sum e_\alpha$ is defined since all but finitely many of the e_α's vanish on a given finite-dimensional subcoalgebra (nonzero orthogonal idempotents are independent). Clearly $e^2 = e$.

2.2.6. Corollary. *Let C be any coalgebra, D a subcoalgebra with $C_0 \subset D$. Suppose $\{f_\alpha\} \subset D^*$ is an orthogonal set of idempotents with $\varepsilon = \sum f_\alpha$. There exists an orthogonal set of idempotents $\{e_\alpha\} \subset C^*$ such that $\varepsilon = \sum e_\alpha$ and $e_\alpha \equiv f_\alpha$ on D for all α.*

Proof. By 2.2.2, there exists an orthogonal set of idempotents $\{e'_\alpha\}$ with $e'_\alpha \equiv f_\alpha$ on D. Fix a particular α, say α_0, and set

$$\begin{aligned} e_\alpha &= e'_\alpha, & \alpha &\neq \alpha_0, \\ &= \varepsilon - \sum_{\alpha \neq \alpha_0} e'_\alpha, & \alpha &= \alpha_0. \end{aligned}$$

Then $\{e_\alpha\}$ is an orthogonal set of idempotents which meets our criteria. Q.E.D.

Now suppose $e, f \in C^*$ are any idempotents and write $e + e' = \varepsilon = f + f'$.

2.2.7. $(e \rightharpoonup C \leftharpoonup f)^\perp = C^* \cdot e' + f' \cdot C^*$, so for subspaces V, $W \subset e \rightharpoonup C \leftharpoonup f$ it follows that $V \wedge W \subset e \rightharpoonup C \leftharpoonup f$.

Proof. That $(e \rightharpoonup C)^\perp = C^* \cdot e'$ and $(C \leftharpoonup f)^\perp = f' \cdot C^*$ is easy to verify. So

$$\begin{aligned} (e \rightharpoonup C \leftharpoonup f)^\perp &= (e \rightharpoonup C \cap C \leftharpoonup f)^\perp \\ &= (e \rightharpoonup C)^\perp + (C \leftharpoonup f)^\perp \\ &= C^* \cdot e' + f' \cdot C^*. \end{aligned}$$

If $I = (e \rightharpoonup C \leftharpoonup f)^\perp$ then $I \subset I^2$ since e' and f' are idempotents. Therefore,

$$(e \rightharpoonup C \leftharpoonup f) \wedge (e \rightharpoonup C \leftharpoonup f) = (I^2)^\perp \subset I^\perp = e \rightharpoonup C \leftharpoonup f. \quad \text{Q.E.D.}$$

2.2.8. If D, E are subcoalgebras of C and $c \in D \wedge E$ then

$$\Delta e \rightharpoonup c \leftharpoonup f \in C \leftharpoonup f \otimes e \rightharpoonup E + D \leftharpoonup f \otimes e \rightharpoonup C.$$

Proof. For $a, b \in C^*$ we have $\Delta a \rightharpoonup c \leftharpoonup b = \sum c_{(1)} \leftharpoonup b \otimes a \rightharpoonup c_{(2)}$, so

$$\begin{aligned} \Delta e \rightharpoonup c \leftharpoonup f &\in (C \otimes E + D \otimes C) \cap C \leftharpoonup f \otimes e \rightharpoonup C \\ &= C \leftharpoonup f \otimes e \rightharpoonup E + D \leftharpoonup f \otimes e \rightharpoonup C. \quad \text{Q.E.D.} \end{aligned}$$

2.2.9. Lemma. *If $e \rightharpoonup C_0 \leftharpoonup e$ is a subcoalgebra then so is $e \rightharpoonup C_1 \leftharpoonup e$; in fact, $e \rightharpoonup C_1 \leftharpoonup e = (e \rightharpoonup C_0 \leftharpoonup e) \wedge (e \rightharpoonup C_0 \leftharpoonup e)$.*

Proof. If $e \rightharpoonup C_0 \leftharpoonup e$ is a subcoalgebra,then $e \rightharpoonup C_0 = e \rightharpoonup C_0 \leftharpoonup e = C_0 \leftharpoonup e$ since C_0 is the direct sum of simple coalgebras. By 2.2.7, $(e \rightharpoonup C_0)$

$\wedge (C_0 \leftharpoonup e) \subset (e \rightharpoonup C \leftharpoonup e) \cap C_1 = e \rightharpoonup C_1 \leftharpoonup e$. By 2.2.8, $e \rightharpoonup C_1 \leftharpoonup e \subset (e \rightharpoonup C_0) \wedge (C_0 \leftharpoonup e)$. Q.E.D.

2.2.10. Corollary. *If $e \rightharpoonup C_0 \leftharpoonup e = f \rightharpoonup C_0 \leftharpoonup f$ and is a coalgebra, then $e \rightharpoonup C_1 \leftharpoonup e = f \rightharpoonup C_1 \leftharpoonup f$.*

Remark. If $e \rightharpoonup C_1 \leftharpoonup e$ is a coalgebra we cannot conclude that $e \rightharpoonup C_2 \leftharpoonup e$ is also in general. The smallest possible example demonstrating this must be at least five dimensional. One such is the following:

Let C have basis 1, z, v, w, and x and define

$$\begin{aligned} \Delta 1 &= 1 \otimes 1, \\ \Delta z &= z \otimes z, \\ \Delta v &= v \otimes z + 1 \otimes v, \\ \Delta w &= w \otimes 1 + z \otimes w, \\ \Delta x &= x \otimes 1 + v \otimes w + 1 \otimes x, \end{aligned} \qquad \begin{aligned} \varepsilon(1) &= 1 = \varepsilon(z), \\ \varepsilon(v) &= \varepsilon(w) = \varepsilon(x) = 0. \end{aligned}$$

If $e(1) = 1$, $e(z) = e(v) = e(w) = e(x) = 0$ then $e^2 = e$ and $e \rightharpoonup C \leftharpoonup e$ is spanned by 1 and x. See also Example 3.4.

2.2.11. Let C be a coalgebra and write $C_0 = \coprod C_x$, where C_x is simple for all x. Then $C_1 = \sum_{x,y} C_x \wedge C_y$.

Proof. For each x we can find in C_0^* idempotents ε_x such that $\varepsilon_x \rightharpoonup C_0 = C_x = C_0 \leftharpoonup \varepsilon_x$. $\{\varepsilon_x\}$ is orthogonal and $\sum \varepsilon_x = \varepsilon$. By 2.2.6, we can find an orthogonal set of idempotents $\{e_x\} \subset C^*$ such that $\sum e_x = \epsilon$ and $e_x \equiv \epsilon_x$ on C_0. Notice

$$C = \bigoplus_{x,y} e_x \rightharpoonup C \leftharpoonup e_y.$$

If $c \in e_x \rightharpoonup C \leftharpoonup e_y$ then $c = e_x \rightharpoonup c \leftharpoonup e_y$, so if $c \in C_1$ in addition, by 2.2.8, $\Delta c \in C \otimes e_x \rightharpoonup C_0 + C_0 \leftharpoonup e_y \otimes C$. Therefore $c \in C_y \wedge C_x$. Q.E.D.

Remarks. Suppose in 2.2.11, $C = C_1$ and is pointed. Write $C_0 = k(X)$ and set $C_x = k \cdot x$ for $x \in X$. If $x \neq y$ and $c \in e_x \rightharpoonup C \leftharpoonup e_y$ then one sees that $\Delta c = c \otimes x + y \otimes c$, and $e_x \rightharpoonup C \leftharpoonup e_y = \{c \in C\colon \Delta c = c \otimes x + y \otimes c\}$. This gives the characterization of pointed coalgebras $C = C_1$ discovered by Taft and Wilson (see [7]).

Combining 2.2.9 with 2.2.7 we have

2.2.12. Let $C = C_1$ and $e^2 = e \in C^*$ such that $e \rightharpoonup C_0 \leftharpoonup e$ is a coalgebra. Then (e) is closed; in fact $(e) = C^* \cdot e + e \cdot C^*$.

Now suppose C is a coalgebra and Rad C^* is nil. If I is an ideal such that Rad(I) is closed, by Proposition 1 on p. 72 of [1] and 3.5.1 of [2], there is an idempotent $e \in I$ which is central (when restricted to C_0) and such that $\text{Rad}(I) = \text{Rad}((e))$. Let $D = (e)^\perp$. If $C = C_1$, we have the exact sequence (u the restriction map)

$$(*) \qquad 0 \to (e) = C^* \cdot e + e \cdot C^* \to C^* \xrightarrow{u} D^* \to 0$$

and $u(I) \subset \operatorname{Rad} D^*$.

2.2.13. Proposition. *Suppose $C = C_1$ is a coalgebra, and I a cofinite ideal with* Rad(I) *closed. Then I contains a cofinite C^*-submodule.*

Proof. $D = D_1$ since $C = C_1$ for any subcoalgebra D of C. The result follows immediately from $(*)$ and 2.1.10 since u is a map of C^*-modules. Q.E.D.

2.2.14. Lemma. *Let C be any coalgebra, $I \subset \operatorname{Rad} C^*$ an ideal. If $I^2 + V$ is a dense subspace of I, then $C^* \cdot V$ is a left ideal dense in I.*

Proof. Let D be a finite-dimensional subcoalgebra and $u: C^* \to D^*$ the restriction map. It suffices to show $u(C^* \cdot V) = u(I)$. But $u(I) = u(I^2 + V) = u(I)^2 + u(C^* \cdot V)$. Thus we may assume $C = D$ and $I^2 + C^* \cdot V = I$. By induction, $I^n + C^* \cdot V = I$ for all n. Since I is nilpotent, $C^* \cdot V = I$. Q.E.D.

2.2.15. Corollary. *Suppose $I \subset \operatorname{Rad} C^*$ is an ideal and $I^2 + V$ is dense in I for some finite-dimensional V. Then I^n is finitely generated as a left ideal (hence closed) and a cofinite subspace of I for all n.*

Proof. By 2.2.14, the closed left ideal $C^* \cdot V$ is dense in I; thus $C^* \cdot V = I$. Suppose $a_1, \ldots, a_r$ span V. Then $I^2 = Ia_1 + \cdots + Ia_r$ implies that I is a finitely generated left ideal of $A = I \oplus k \cdot \varepsilon$. The rest follows by 1.1.1 of [2]. Q.E.D.

It should be noted that if $C = C_n$ for some n then the ideal in 2.2.15 is in fact finite dimensional.

2.2.16. Proposition. *Suppose $C = C_1$ and I is an ideal of C^* such that* Rad(I) *is closed. If $I^2 + V$ is dense in I for some finite-dimensional V, then $I = C^* \cdot a_1 + \cdots + C^* \cdot a_n + b \cdot C^*$ for some $a_1, \ldots, a_n, b \in I$.*

Proof. Consider $(*)$. $u(I)$ satisfies the hypothesis of the proposition. By 2.2.15, $u(I)$ is finite dimensional. The conclusion now quickly follows. Q.E.D.

2.2.17. Proposition. *Suppose $C = C_1$ and I is an ideal of C^* with* Rad(I) *cofinite. If I contains a finitely generated dense ideal then $I = C^* \cdot a_1 + \cdots + C^* \cdot a_n + b \cdot C^*$ for some $a_1, \ldots, a_n, b \in C^*$.*

Proof. By 2.1.7, Rad(I) is closed. First assume C is almost connected and $I \subset \operatorname{Rad} C^*$. Let $A = \operatorname{Rad} C^* \oplus k \cdot \varepsilon$. Since $\dim(C^*/A) < \infty$ the dense ideal L of I is generated as an ideal of A by $a_1, \ldots, a_r$, say. Since A is commutative $L = A \cdot a_1 + \cdots + A \cdot a_r = \bar{L}$ which implies $L = I$. Now we use $(*)$ to complete the proof. Q.E.D.

We can generalize 2.2.16 to any coalgebra. The conclusion, however, will not be quite as sharp.

2.2.18. Proposition. *Let C be any coalgebra and I an ideal of C^* with* Rad(I) *closed. If $I^2 + V$ is dense in I for some finite-dimensional V, then I^n contains a finitely generated dense ideal and $\overline{I^n}$ is a cofinite subspace of $\bar{I}$ for all n.*

Proof. By 3.5.1. of [2] and 2.2.3, $J = \text{Rad}(\bar{I}) = \text{Rad}(I) = \text{Rad}((e))$ for some idempotent e. If $D = (e)^{\perp}$ we have the exact sequence $0 \to \overline{(e)} \to C^* \to^u D^* \to 0$, where u is the restriction map, and $u(J) = \text{Rad}\, D^*$. But $e \in J$ implies $e \in \bigcap J^n \subset \bar{I}$, so $\overline{(e)} \subset \overline{I^n} = \overline{I^n}$ for all n. By 2.2.14, $\overline{I^n} = u^{-1}(u(I^n))$ is a cofinite subspace of $\bar{I} = u^{-1}(u(I))$. The remaining part will follow once we observe that if $a \in I$ and $u(a) = u(e)$ then $\overline{(e)} \subset \overline{(a^n)}$ for all n. Q.E.D.

If $I \subset C^*$ is an ideal then $\text{Rad}(\bar{I})$ is closed by 2.1.6. Thus

2.2.19. Corollary. *Suppose C is any coalgebra, I an ideal of C^* such that $I^2 + V$ is dense in I for some finite-dimensional V. Then $\overline{I^n}$ is an ideal cofinite in $\bar{I}$ for all n.*

3. **Locally finite coalgebras.** It has been shown in [2] that C reflexive implies C is locally finite and the coradical C_0 is reflexive. The converse is true in the commutative case, but even the hypothesis that the second term of the coradical filtration C_1 is reflexive does not as much imply that C_2 is reflexive in general. We find in this section a topological description of locally finite coalgebras with C_0 reflexive which gives the commutative theorem as a special case. First we need partial generalization of 3.2.1 of [1].

3.1. Lemma. *Suppose $C = C_1$ and is almost connected. If all cofinite C^*-modules of C^* are closed, then C is finite dimensional.*

Proof. All maximal ideals of C^* are closed since $\text{Rad}\, C^* = C_0^{\perp}$ is cofinite. Since cofinite C^*-modules are closed by assumption, by 2.1.12 every cofinite ideal of C^* is closed. Thus C is reflexive. By 3.2.1 of [2], C is finite dimensional. Q.E.D.

3.2. Theorem. *Let C be any coalgebra. Then the following are equivalent:*

(1) *C_0 is reflexive and C is locally finite.*

(2) *All cofinite C^*-modules of C^* are closed.*

(3) *Every cofinite ideal of C^* contains a finitely generated dense ideal.*

Proof. (1) ⇒ (2). Let I be a cofinite C^*-module of C^*. Then $\text{Rad}(I) = J$ is closed since C_0 is reflexive; so if $C^* \to^u C_0^*$ is the projection we conclude $u(J) = (u(a))$ for some $a \in I$. Let $D = (a)^{\perp}$. Then $D_0 = J^{\perp}$ and is therefore finite dimensional. Since C is locally finite D_1 is finite dimensional, and this implies D is strongly reflexive by 4.1.1 of [2]. Since $C^* \cdot a = \overline{(a)}$ we have the exact sequence $0 \to C^* \cdot a \to C^* \to^{\pi} D^* \to 0$. But $\pi(I)$ is finitely generated as a left ideal of D^* so, by 2.1.4, $I = \pi^{-1}(\pi(I))$ is a finitely generated C^*-module. This means that I is closed.

(2) ⇒ (3). Let I be a cofinite ideal of C^*. Then $C^* \cdot I = E^{\perp}$ where E is a finite-dimensional subcoalgebra of C. If $D = E_0 \wedge E_0$, then $D_0 = E_0$ and, using 2.1.4, we see the cofinite D^*-modules of D^* are closed. By 3.1, $D = D_1$ is finite dimensional. Since $D \wedge D \subset \wedge^n D_0$ some n we have that $D \wedge D$ is finite dimensional by 4.2.2 of [2]. By the same theorem $C^* \cdot I$ is a finitely generated submodule. It is easy to find generators in I. Since $C^* \cdot I = \bar{I}$, we are done.

(3) ⇒ (1). By 2.1.5 all cofinite maximal ideals of C^* are closed, so C_0 is reflexive

by 3.5.3 of [2]. By 4.2.2 of [2], C is locally finite. Q.E.D.

If $C = C_1$ we can make a much stronger statement.

3.3. Corollary. *Suppose $C = C_1$. Then the following are equivalent:*

(1) *C_0 is reflexive and C is locally finite.*

(2) *C is reflexive.*

(3) *If I is a cofinite ideal of C^*, then $I = C^* \cdot a_1 + \cdots + C^* \cdot a_n + b \cdot C^*$ for some $a_1, \ldots, a_n, b \in I$.*

Proof. (1) $\Leftrightarrow$ (3) follows by 3.2 and 2.2.17.

(3) $\Rightarrow$ (2) is clear since principal left (or right) ideals of C^* are closed.

(2) $\Rightarrow$ (1) is clear. Q.E.D.

Remarks. (1) If C is commutative, then $C^* \cdot a = (a)$ so Theorem 3.2 gives the essence of 5.1.1 of [2].

(2) If C_0 is reflexive (which one may reasonably expect for infinite fields k; see §3.7 of [2]) then 3.2 gives a topological formulation of local finiteness. C_0 reflexive and C locally finite does not necessarily imply C_2 reflexive. This means 3.3 cannot be improved in general.

3.4. Example. Let k be an infinite field, and let N denote the set of positive integers. For $1 < n \in N$ choose vector spaces $V(n) \simeq k^n \simeq W(n)$, set $V = \bigoplus V(n)$, $W = \bigoplus_n W(n)$ and $V \cdot W = \bigoplus_n V(n) \otimes W(n)$ (let $v \cdot w = v \otimes w$ for $v \in V(n)$ and $w \in W(n)$). Endow $C = N \oplus V \oplus W \oplus V \cdot W$ with the coalgebra structure determined by

$$\Delta n = n \otimes n, \qquad \varepsilon(n) = 1 \quad \text{for all } m \in N,$$

$$\Delta v = v \otimes n + 1 \otimes v, \qquad \varepsilon(v) = 0,$$

$$\Delta w = w \otimes 1 + n \otimes w, \qquad \varepsilon(w) = 0,$$

$$\Delta v \cdot w = v \cdot w \otimes 1 + v \otimes w + 1 \otimes v \cdot w, \qquad \varepsilon(v \cdot w) = 0 \quad \text{for all } v \in V(n) \text{ and } w \in W(n).$$

One can check easily that $C_0 = k^{(N)}$, so is reflexive by 3.6 of [4]; and that $C_1 = N \oplus V \oplus W$ and is locally finite, so C is also.

Let $\mathcal{M} = 1^\perp$. Then $\mathcal{M}$ is a maximal ideal of C^*. The functionals $\mathcal{M}^2$ restricted to $V \cdot W$ is a proper dense subspace. Thus $\mathcal{M}^2 \subsetneq \mathcal{M}$ and $\mathcal{M}^2$ is *not* closed. Since $C^* = k \cdot \varepsilon \oplus \mathcal{M}$, any subspace I satisfying $\mathcal{M}^2 \subset I \subset \mathcal{M}$ is an ideal. Therefore there are cofinite ideals I which are not closed satisfying $\mathcal{M}^2 \subset I \subset \mathcal{M}$. This means C is not reflexive.

Remarks. In the preceding example $k \cdot 1 \wedge k \cdot 1 = k \cdot 1$. Suppose $e \in C^*$ is *any* idempotent satisfying $e(n) = 1 - \delta_{1,n}$. Then

(1) $\overline{(e)} = \mathcal{M}$, so (e) is not closed since $(e) \subset \mathcal{M}^2$;

(2) if $\mathcal{M}^2 \subset I \subset \mathcal{M}$, then $e \in I$; therefore $\overline{(e)} = \bar{I}$. Thus a cofinite ideal which contains a finitely generated dense ideal is not necessarily closed.

4. The closure of an ideal. Example 3.4 shows that in general a cofinite ideal of C^* which contains a finitely generated dense ideal is not closed. Proposition 4.3

gives a condition under which such an ideal is closed. From this result we derive the main theorem of this section, which concerns the role Rad $\mathcal{C}^*$ plays in the description of the closure of a ideal.

The first lemma is a special case of 4.3.

4.1. Lemma. *Suppose C is connected and I is an ideal of C^* which contains a finitely generated dense ideal. If $C/I^\perp$ is a semisimple $\mathcal{C}$-comodule, then I is finite dimensional.*

Proof. We have the commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & \mathcal{C} \otimes C \\ \big\downarrow & & \big\downarrow \\ C/D & \longrightarrow & \mathcal{C} \otimes C/D \end{array}$$

where $D = I^\perp$. If $G(C) = \{1\}$ then $\mathcal{C}_0 = k \cdot 1 \otimes 1$. By 1.2.3, C/D is in fact a $\mathcal{C}_0$-comodule. Thus if $(C/D, \omega)$ is the underlying $\mathcal{C}$-comodule structure on C/D, then $\omega(\overline{c}) = 1 \otimes 1 \otimes \bar{c}$ for all $c \in C$. This means $\sum c_{(1)} \otimes c_{(3)} \otimes c_{(2)} - 1 \otimes 1 \otimes c \in \ker \pi = C \otimes C \otimes D$ or

$$\sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)} - 1 \otimes c \otimes 1 \in C \otimes D \otimes C \quad \text{for } c \in C.$$

If $f \in I$ then $f(D) = (0)$; so, given $a, b \in C^*$,

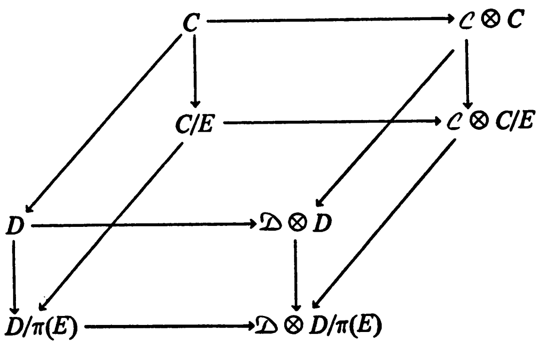
$$\langle a * f * b, c \rangle = \langle a * b, 1 \rangle \langle f, c \rangle \quad \text{for each } c \in C$$

which implies $a * f * b = \langle a * b, 1 \rangle f$. Therefore any subspace of I is an ideal, so I is finite dimensional by assumption. Q.E.D.

To reduce 4.3 to the connected case we will need the following technical lemma.

4.2. Lemma. *Let D be connected and π: $C \to D$ a surjective coalgebra map with $\pi(C_0) = D_0$. If E is a subcoalgebra of C and C/E is a semisimple $\mathcal{C}$-comodule then $D/\pi(E)$ is a semisimple $\mathfrak{D}$-comodule.*

Proof. We have the array



with commuting top and sides, hence the bottom diagram commutes. Now $\mathcal{C}_0 \subset C_0 \otimes C_0$ so if $\Pi\colon \mathcal{C} \to \mathfrak{D}$ is the induced coalgebra map, $\Pi(\mathcal{C}_0) \subset \pi(C_0) \otimes \pi(C_0) = \mathfrak{D}_0$ since D is connected. Therefore $\Pi^*(\text{Rad}\, \mathfrak{D}^*) \subset \text{Rad}\, \mathcal{C}^*$. Using the bottom of the above array we have the commutative diagram

$$\begin{array}{ccccc} C/E & \longrightarrow & \mathcal{C} \otimes C/E & \xrightarrow{\Pi^*(f) \otimes I} & C/E \\ \downarrow & & \downarrow & & \downarrow \\ D/\pi(E) & \longrightarrow & \mathfrak{D} \otimes D/\pi(E) & \xrightarrow{f \otimes I} & D/\pi(E) \end{array}$$

for any $f \in \mathfrak{D}^*$. Therefore, if $f \in \text{Rad}\, \mathfrak{D}^*$, by 1.2.4, $\bar{d} \leftharpoonup f = 0$ for all $d \in D$. This means again by 1.2.4 that $D/\pi(E)$ is semisimple. Q.E.D.

4.3. Proposition. *Let C be a coalgebra, I an ideal of C^* with* Rad(I) *cofinite and which contains a finitely generated dense ideal. If $C/I^\perp$ is a semisimple $\mathcal{C}$-comodule, then I is closed.*

Proof. Let $C \xrightarrow{\pi} C/D = N$ be the projection where $D = I^\perp$. Write $C_0 = E_1 \oplus D_0$. Since N is semisimple, $\pi(E_1) \oplus M = N$ for some subcomodule M of N. By 2.1.2, $E_2 = \pi^{-1}(M)$ is a subcoalgebra of C, and

$$E_1 \cap E_2 = E_1 \cap D = E_1 \cap D_0 = (0).$$

Therefore $C^* = E_1^* \times E_2^*$. Let $u\colon C^* \to E_2^*$ be the restriction map. Now $J = \text{Rad}(I)$ is closed by 2.1.7 so $\ker u \subset J$ since $J^\perp = D_0 \subset E_2$. From this it is easy to see that $\ker u \subset I$. Therefore $I = u^{-1}(u(I))$. Clearly $u(I)$ satisfies the hypothesis of the proposition. The map $E_2 \xrightarrow{\pi} C/D$ is a map of comodules; therefore $M = \pi(E_2)$ is a semisimple $\mathcal{E}_2$-comodule. Hence to prove the proposition we may assume $C = E_2$. But also notice $u(I)^\perp = I^\perp = D$ and

$$(E_2)_0 = C_0 \cap E_2 = (E_1 \oplus D_0) \cap E_2 = D_0 \cap E_2 = D_0.$$

Thus

4.4. To prove 4.3 we may assume in addition that C is almost connected and $I \subset \text{Rad}\, C^*$.

Now assume the additional hypothesis. If we can reduce to the connected case we will be done by 4.1.

Let $\overline{C} = R(C) = C/C_0^+$ and $C \xrightarrow{\rho} \overline{C}$ be the projection. Suppose $\rho^* = f\colon \overline{C}^* \to C^*$ is the induced algebra map; set $\gamma = f^{-1}(I)$ and $\mathcal{L} = f^{-1}(L)$ where $L \subset I$ is a finitely generated ideal dense in I. Then $I = f(\gamma)$ and $L = f(\mathcal{L})$ since $f(\overline{C}^*) = \text{Rad}\, C^* \oplus k \cdot 1$. Now f is continuous and linearly closed so

$$f(\overline{\mathcal{L}}) = \overline{f(\mathcal{L})} = \overline{L} = \overline{I} = f(\overline{\gamma})$$

which implies that $\mathcal{L}$ is dense in γ. Since $\dim(C^*/\overline{C}^*) < \infty$ we have $C^* \otimes C^{*\text{op}}$ is a finitely generated left $\overline{C}^* \otimes \overline{C}^{*\text{op}}$-module. This means γ is a finitely generated ideal of $\bar{C}^*$. To show $I = f(\gamma)$ is closed we need only show that γ is closed. Since $\rho(I^\perp) = \gamma^\perp$ by 4.2, $\bar{C}/\gamma^\perp$ is a semisimple $\bar{\mathcal{C}}$-comodule. Thus we may assume $C = \bar{C} = R(C)$ and the reduction to the connected case is complete. Q.E.D.

Now we come to the main result of this section. The proof is a reduction to the previous proposition.

4.5. Theorem. *Suppose C is a coalgebra, I an ideal of C^* with* Rad(I) *cofinite and which contains a finitely generated dense ideal. Then $\mathcal{R} \cdot I = \bar{I}$.*

Proof. The projection $\pi\colon C \to C/I^\perp = N$ is a surjection of $\mathcal{C}$-comodules since $I^\perp$ is a subcoalgebra. Since $\pi^*(N^*) = \bar{I}$ we see that N^* is a finitely generated $\mathcal{C}^*$-module since I contains a finitely generated dense ideal.

Set $D = \pi^{-1}(N_0)$. Then D is a subcoalgebra of C.

We derive from the commutative diagram

$$\begin{array}{ccc} N_0 & \longrightarrow & N \\ \uparrow & & \uparrow \\ D & \longrightarrow & C \end{array}$$

the commutative diagram

$$\begin{array}{ccccccccc} & & & & C^* & \longrightarrow & D^* & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{J} \cdot N^* & \longrightarrow & N^* & \longrightarrow & N_0^* & \longrightarrow & 0 \end{array}$$

with bottom row exact by 1.2.5. Since $\mathcal{J} \cdot I = \mathcal{J} \cdot (\mathcal{C}^* \cdot I) = \mathcal{J} \cdot \bar{I}$ we have the commutative diagram

$$\begin{array}{ccccccccc} & & & & C^* & \xrightarrow{f} & D^* & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{J} \cdot I & \longrightarrow & \bar{I} & \xrightarrow{g} & N_0^* & \longrightarrow & 0, \end{array}$$

where f is the restriction map.

Now $\mathcal{R} \cdot I = g^{-1}(g(\bar{I}))$. Therefore to show $\mathcal{R} \cdot I = \bar{I}$ we need only show $g(I) = g(\bar{I})$, or equivalently $f(I) = f(\bar{I})$. Since f is continuous and linearly closed $f(\bar{I}) = \overline{f(I)}$. Thus it suffices to show that $f(I)$ is closed. It is clear $f(I)$ is an ideal of D^* which satisfies the hypotheses of the theorem. Thus we may assume $C = D$. Observe $N_0 = \pi(D) = D/I^\perp$ is a semisimple $\mathcal{D}$-comodule, since $\pi\colon D \to N_0$ is a surjective $\mathcal{C}$-comodule map. Since $f(I)^\perp = I^\perp$ we are reduced to Proposition 4.3. Q.E.D.

4.6. Corollary. *Let I be a cofinite ideal of C^* which contains a finitely generated dense ideal. Then $\bar{I} = u_1 \cdot I + \cdots + u_\gamma \cdot I$ for some units $u_1, \ldots, u_\gamma$ of $\mathcal{R}$.*

For any ideal I of C^* let $\underline{I}$ denote the sum of the $\mathscr{R}$-submodules of I. Then for $a \in \underline{I}$ the computation $\mathscr{R} \cdot (a) = (a) + \mathcal{J} \cdot a$ shows that $\underline{I}$ is also an ideal.

4.7. Corollary. *Let I be a cofinite ideal of C^* which contains a finitely generated dense ideal. Then I is closed if and only if* $\operatorname{Rad}(\underline{I})$ *is cofinite and closed.*

Proof. ($\Rightarrow$) This is clear since $\operatorname{Rad}(\underline{I}) = \operatorname{Rad}(I)$ is closed in this case by 2.1.7. ($\Leftarrow$) Suppose $\operatorname{Rad}(\underline{I})$ is cofinite and closed. Let $u\colon C^* \to C_0^*$ be the projection. Then $u(\underline{I}) = u(\operatorname{Rad}(\underline{I}))$, so $u(\underline{I}) = u((a))$ for some $a \in \underline{I}$. But $\operatorname{Rad}((a)) = \operatorname{Rad}(\underline{I})$ which implies $\overline{(a)} = \mathscr{R} \cdot (a) \subset \underline{I}$ by 4.5. Let $D = (a)^{\perp}$. Then $D_0 = \operatorname{Rad}(I)^{\perp}$ is finite dimensional and $0 \to \mathscr{R} \cdot (a) \to C^* \xrightarrow{\pi} D^* \to 0$ is exact. D is reflexive (see 4.2.6 of [2]) so $I = \pi^{-1}(\pi(I))$ is closed. Q.E.D.

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